Supplement to
“An Axiomatization of Plays in Repeated Games”

Laurent Mathevet *

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Abstract

We classify all stage games, and indirectly all repeated games, into families based on the degree of sophistication required to satisfy the strong axiom of efficient simplicity.

1 Classification of Games

For $G \in \mathcal{G}$, $\Pi(G) = \{\pi \in \mathbb{R}^2 : \text{there is } a \in A \text{ s.t. } \pi = (u_1(a), u_2(a))\}$ is the set of pure payoff profiles.

A triangle is a convex set $\Delta = \text{Co}(\{u^1, u^2, u^3\})$ where u^1, u^2 and u^3 are three (non-collinear) payoffs in $\Pi(G)$,[1] called *vertices*. Let $V(\Delta)$ be the set of vertices of Δ. For any $X \subset \mathbb{R}^2$, let

$$\mathcal{P}(X) = \{\pi \in X : \nexists x \in X \text{ s.t. } x \gg \pi\}$$

be the set of weakly Pareto efficient points in X relative to X. Let $\mathcal{N}(\Delta) = \{u \in V(\Delta) : u \in \mathcal{P}(V(\Delta))\}$ be the set of vertices of Δ that are pairwise Pareto undominated. Let $\mathcal{T}(G)$ be the set of all triangles in G.

Given a game $G \in \mathcal{G}$, the next definition classifies the areas of the feasible payoffs into sets. This will determine the family to which the game belongs. Let $\mathbb{Q}[0, 1]$ be

*Dept. of Economics, New-York University, 19 W. 4th Street, 6FL, New York, NY 10012, U.S.A. (lmath@nyu.edu)

[1] Three points $x, y, z \in \mathbb{R}^2$ are collinear if $\frac{y_2 - x_2}{y_1 - x_1} = \frac{z_2 - x_2}{z_1 - x_1}$, i.e., if they are on a straight line.

the set of rational numbers between 0 and 1. Given a rational number q, $d(q)$ denotes its denominator and $\nu(q)$ its numerator. In the definition, k is a natural number. Let Co denote the convex hull operator.

Definition 1. *A payoff* $\pi \in Co(\Pi(G))$ *is in* $\mathcal{U}_k$ *if for all triangles* $\Delta \in \mathcal{T}(G)$ *such that* $\pi \in \Delta$ *and* $|\mathcal{N}(\Delta)| = 3$*, we can write* $\Delta = Co(\{u^1, u^2, u^3\})$ *such that, for some* $q \in \mathbb{Q}[0,1]$*, either* $(u^2 - u^1) + q(u^2 - u^3) \gg 0$ *and* $d(q) \leq k$*, or* $(u^1 - u^2) + q(u^3 - u^2) \gg 0$ *and* $d(q) + \nu(q) \leq k$.[2] *If there is no triangle containing* π *such that* $|\mathcal{N}(\Delta)| = 3$*, then* $\pi \in \mathcal{U}_1$.

A feasible payoff is in $\mathcal{U}_k$ if every triangle that contains it, and whose vertices are pairwise Pareto unordered, can be written such that a movement along one of its edges, followed by a fraction q of a movement along another edge, leads to a Pareto improvement, where q satisfies certain properties with respect to k. Once all feasible payoffs are placed into some $\mathcal{U}_k$, the largest of these k values is the family to which G belongs.

Definition 2. *A game* $G \in \mathcal{F}_n$*, where* $n \in \mathbb{N}_+$*, if for every* $\pi \in Co(\Pi(G))$*,* $\pi \in \cup_{k=1}^{n} \mathcal{U}_k$.

Definition 3. *A game* $G \in \mathcal{F}_1^*$*, if for all* $\Delta \in \mathcal{T}(G)$ *and* $\pi \in \Delta$*, either (i)* $|\mathcal{N}(\Delta)| = 1$ *or (ii)* $|\mathcal{N}(\Delta)| = 2$ *and* $\Delta = Co(\{u^1, u^2, u^3\})$ *such that* $u^2 \gg u^1$ *and* $u^3 \gg u^1$.

Every stage game belongs to a family $\mathcal{F}_n$ and every family $\mathcal{F}_n$ is nonempty.

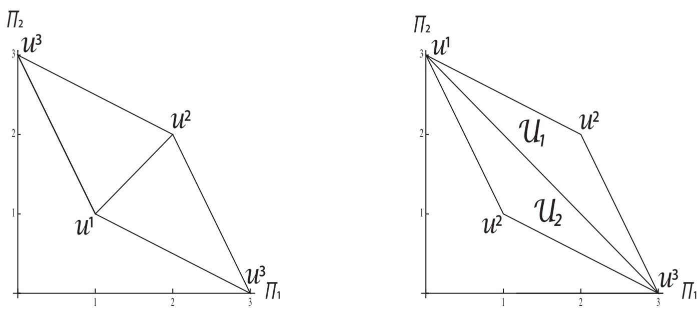

Figure 1: Triangles in the Prisoners' Dilemma

[2] The requirement $q \in [0,1]$ is inconsequential because we can relabel the vertices of a triangle: if $(u^2 - u^1) + 2(u^2 - u^3) \gg 0$, then $\frac{1}{2}(u^2 - u^1) + (u^2 - u^3) \gg 0$; exchanging the labels of u^1 and u^3 gives the desired form.

In the Prisoners' Dilemma of Figure 3 (p.18), there are four possible triangles, as shown by Figure 1. For every triangle in the left panel, $|\mathcal{N}(\Delta)| = 2$ and thus, if these were the only triangles, the game would be in $\mathcal{F}_1$. However, triangles in the right panel satisfy $|\mathcal{N}(\Delta)| = 3$. In the bottom triangle, a movement along $(u^1 - u^2) + (u^3 - u^2)$ leads to a Pareto improvement. Therefore, $q = 1$ and hence $d(q) + v(q) = 2$. In the top triangle, a movement along $(u^2 - u^1) + (u^2 - u^3)$ gives a Pareto improvement. Therefore, $q = 1$ and hence $d(q) = 1$. Putting these observations together, we obtain $G \in \mathcal{F}_2$.

Battle of the Sexes and Chicken are in $\mathcal{F}_1$. As shown by Figure 3 of Section 5, the set of feasible payoffs in Battle of the Sexes has only one triangle and it satisfies $|\mathcal{N}(\Delta)| = 2$. Thus, Battle of the Sexes is in $\mathcal{F}_1$. For Chicken, only one triangle satisfies $|\mathcal{N}(\Delta)| = 3$, and we can label its vertices so that $(u^2 - u^1) + (u^2 - u^3) \gg 0$. Thus, Chicken is in $\mathcal{F}_1$.

Battle of the Sexes, Stag Hunt, the Ultimatum Game, Common Interest, and the Samaritan's Dilemma are actually in $\mathcal{F}_1^*$.

To generate further examples and assist the reader with this classification, we provide an online program that classifies 2×2 games.[3]

2 Result

According to Definitions 1 to 3, we can classify all stage games into families $\mathcal{F}_n$, where $n \geq 1$ is a natural number, based on geometric properties of their set of feasible payoffs. Suppose that we have done so. Then, the following proposition formalizes the relationship between the axiom and complexity.

Proposition 1.
(a) For any game $G \in \mathcal{F}_n$, if a convention s contains a subsequence s' such that $\pi(s') \gg \pi(s)$, then there is a subsequence s'' of s such that s'' is at most n times more complex than s and $\pi(s'') \gg \pi(s)$.

(b) For any game $G \in \mathcal{F}_1^*$, if a deterministic convention s contains a subsequence s' such that $\pi(s') \gg \pi(s)$, then there is a deterministic subsequence s'' of s that is strictly simpler than s and such that $\pi(s'') \gg \pi(s)$.

[3] The program is available at https://files.nyu.edu/lm2737/public/research.html. The user specifies the payoffs and the program outputs $\mathcal{F}_n$ if $n < 10,000$. I thank Jonathan Lhost for creating this program. The reader can check that $u(c, c) = u(d, d) = (6, 1)$, $u(c, d) = (1, 8)$, and $u(d, c) = (2, 7)$ is in $\mathcal{F}_5$.

This proposition states that for stage games in $\mathcal{F}_n$, if a social convention s has a strictly Pareto-improving subsequence, then s also has one that is less than n times more complex than s.

Proof of Proposition 1. **(a)**. Take any convention s. Consider two cases: (a) $|R(s)| \leq 2$, (b) $|R(s)| \geq 3$.

Case (a). If $|R(s)| = 1$, there is no subsequence s' of s such that $\pi(s') \gg \pi(s)$ and this violates the assumption. If $|R(s)| = 2$, then either the elements of $R(s)$ are not Pareto ranked, i.e., $R(s) = \{u, u'\}$ with $u \not\gg u'$ and $u' \not\gg u$, in which case the assumption is violated because there is no subsequence s' of s such that $\pi(s') \gg \pi(s)$; or these elements are Pareto ranked (say $u \gg u'$, without loss of generality). Since a Pareto-improving subsequence exists, u' must be played with strictly positive frequency. Hence the subsequence s'' of s, which always plays the profile yielding u, strictly dominates s (if s is cyclic, then s'' is also strictly simpler than s, since s'' has a cycle of length 1).

Case (b). Suppose $|R(s)| \geq 3$. If $\pi(s) \in \mathcal{P}(\text{Co}(R(s)))$, then there is no subsequence s' of s such that $\pi(s') \gg \pi(s)$, which violates the assumption. Therefore, suppose $\pi(s) \notin \mathcal{P}(\text{Co}(R(s)))$. Then there must be a triangle $\Delta = \text{Co}(\{u^1, u^2, u^3\})$ such that (i) $\pi(s) \in \Delta$, (ii) $\{u^1, u^2, u^3\} \subset R(s)$, and (iii) $\pi(s) \notin \mathcal{P}(\Delta)$. There are two sub-cases:

Case (b1). Suppose $|\mathcal{N}(\Delta)| \leq 2$. This implies that there exist $u, u' \in \{u^1, u^2, u^3\}$ such that $u \ll u'$ and u is played with strictly positive frequency in s. Let a' be an action profile that appears infinitely often in s and for which $u(a') = u'$. Construct the subsequence s'' of s by transferring the frequency attached to all profiles yielding u to a'. Clearly, $\pi(s'') \gg \pi(s)$ and $\ell(s'') \leq \ell(s)$.

Case (b2). Suppose $|\mathcal{N}(\Delta)| = 3$. Since s has convergent frequencies, we can write $\pi(s)$ as

$$\pi(s) = \sum_{m=1}^{M} \frac{w_m(s)}{\ell(s)} u^m$$

where $M \geq 3$, $\{u^m\} \subset R(s)$, $w_m(s) \geq 0$ for all m, and $\ell(s) = \sum_{m=1}^{M} w_m(s)$. Recall that $\pi(s) \in \Delta = \text{Co}(\{u^1, u^2, u^3\})$. For any $w^* > 0$, define

$$\pi^* = \pi(s) + \frac{w^*}{\ell(s)}((u^2 - u^1) + q(u^2 - u^3)) \tag{1}$$

and

$$\pi^{**} = \pi(s) + \frac{w^*}{\ell(s)}((u^1 - u^2) + q(u^3 - u^2)), \tag{2}$$

where $q \in \mathbb{Q}[0,1]$ is written as $q = \nu(q)/d(q)$. Since G is in family $\mathcal{F}_n$, either $(u^2 - u^1) + q(u^2 - u^3) \gg 0$ and $d(q) \leq n$, or $(u^1 - u^2) + q(u^3 - u^2) \gg 0$ and $d(q) + \nu(q) \leq n$. Thus, for all $w^* > 0$, either $\pi^* \gg \pi(s)$ or $\pi^{**} \gg \pi(s)$ holds. In each case, we can choose w^* to obtain a well-defined convex combination. For $w^* = \min\{w_1(s), w_3(s)\}$, (1) gives

$$\pi^* = \sum_{m \neq 1,2,3} \frac{w_m(s)d(q)}{d(q)\ell(s)} u^m + \frac{(w_1(s) - w^*)d(q)}{d(q)\ell(s)} u^1 + \frac{w_2(s)d(q) + w^*(\nu(q) + d(q))}{d(q)\ell(s)} u^2 + \frac{w_3(s)d(q) - w^*\nu(q)}{d(q)\ell(s)} u^3,$$

which is a well-defined convex combination implementable by a sequence of length at most $d(q)\ell(s)$ (which is less than $n\ell(s)$). For $w^* = w_2(s)/(1+q)$, (2) gives

$$\pi^{**} = \sum_{m \neq 1,2,3} \frac{w_m(s)(d(q) + \nu(q))}{\ell(s)(d(q) + \nu(q))} u^m + \left(\frac{w_1(s)(d(q) + \nu(q)) + d(q)w_2(s)}{\ell(s)(d(q) + \nu(q))}\right) u^1 + \left(\frac{w_3(s)(d(q) + \nu(q)) + \nu(q)w_2(s)}{\ell(s)(d(q) + \nu(q))}\right) u^3 \tag{3}$$

which is a well-defined convex combination implementable by a sequence of length at most $(d(q) + \nu(q))\ell(s)$ (which is less than $n\ell(s)$). This completes the proof of part 1.

(b) The proof follows similar steps to part 1. Case (a) has already been addressed and only Case (b1) applies. Case (b1) described a procedure that delivers a Pareto-improving subsequence (by abandoning some action profiles). However, the resulting subsequence might also be Pareto improved in a similar fashion. Therefore, this procedure can be used iteratively by eliminating profiles in $R(s)$, and it produces one Pareto-improving subsequence after another without increasing complexity along the way. Since A is finite, there must be a point at which the sequence under consideration, call it s, satisfies $R(s) = \{u^1, u^2, u^3\}$, i.e., only three payoff profiles appear in s. From here, the next argument shows that a strictly simpler Pareto-improving subsequence can be found. Let $A^i \subset A$ be the set of action profiles that appear infinitely often in s such that $u(a^i) = u^i$ for all $a^i \in A^i$. Define $\Delta = \mathrm{Co}(\{u^1, u^2, u^3\})$. If $|\mathcal{N}(\Delta)| = 1$, then there is $u^i \in \{u^1, u^2, u^3\}$ such that $u^i \gg u^j$ for $j \neq i$. Thus, the constant sequence s'' in which players always play $a^i \in A^i$ satisfies $\pi(s'') \gg \pi(s)$ and it is a subsequence of s.

Given s is cyclic and $|R(s)| = 3$, s'' must be strictly simpler than s, because $\ell(s'') = 1$. Now suppose $|\mathcal{N}(\Delta)| = 2$. It follows from the definition of $\mathcal{F}_1^*$ that $u^2 \gg u^1$ and $u^3 \gg u^1$. We construct a subsequence s'' of s and then show that it is Pareto-improving and strictly simpler than s. Let $W_i(s) = \sum_{a \in A^i} w_a(s)$ be the weight assigned to payoff u^i in s where $\ell(s) = W_1(s) + W_2(s) + W_3(s)$ is the length of the cycle of s. Construct s'' from s as follows: (1) Pick $a^* \in A^2$ and $a^{**} \in A^3$; a^* and a^{**} will be the only profiles in $A^2 \cup A^3$ that are played in s''; (2) No profile in A^1 is played in s''; (3) a^* is played with frequency

$$\frac{W_2(s) + \frac{W_2(s)}{W_2(s)+W_3(s)} W_1(s)}{\ell(s)};$$

(4) a^{**} is played with frequency

$$\frac{W_3(s) + \frac{W_3(s)}{W_2(s)+W_3(s)} W_1(s)}{\ell(s)}.$$

In words, s'' re-allocates all of $W_2(s)$ to a^*, all of $W_3(s)$ to a^{**}, and all of $W_1(s)$ to a^* and a^{**}. Since $u^2 \gg u^1$ and $u^3 \gg u^1$, s'' is necessarily Pareto-improving. Simple calculations show that a^2 is played with frequency $W_2(s)/(W_2(s) + W_3(s))$ and a^3 with frequency $W_3(s)/(W_2(s) + W_3(s))$. Hence, $\ell(s'') = W_2(s) + W_3(s) < \ell(s)$. □